Lecture Notes: Functional Analysis

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1 Basic Concepts

1.1 Algebra

A collection $\mathcal{B}$ of subsets of X is called an **algebra** of sets if for every A and B in $\mathcal{B}$, $A \cap B$, $A \cup B$ and the complement of A are in $\mathcal{B}$.
If in addition, every countable union of sets in $\mathcal{B}$ is still in $\mathcal{B}$, then $\mathcal{B}$ is called a σ-algebra.

1.2 Upper and Lower Limits

Let $\{x_n\}$ be a sequence of real numbers. Then the upper and lower limits (limit superior and limit inferior) of $\{x_n\}$ are

$$\overline{\lim} x_n = \inf_n \sup_{k \geq n} x_k = \sup\{y | \lim_{j \to \infty} x_{n_j} = y \text{ for some subsequence } \{x_{n_j}\}\}$$

$$\underline{\lim} x_n = \sup_n \inf_{k \geq n} x_k = \inf\{y | \lim_{j \to \infty} x_{n_j} = y \text{ for some subsequence } \{x_{n_j}\}\}.$$

1.3 Continuous Functions

- A function f is continuous if and only if any of the following holds:
  (1) For every x and $\epsilon > 0$, there is $\delta > 0$ such that for every y, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
  (2) For every x, and every sequence $\{x_n\}$ such that $x_n \to x$, we have $f(x_n) \to f(x)$.
  (3) The pre-image of every open set is open.
  (4) The pre-image of every closed set is closed.
- A function f is uniformly continuous if and only if for every $\epsilon > 0$, there is $\delta > 0$ (which does not depend on x) such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

1.4 Borel Sets

- The smallest σ-algebra containing all open sets (or closed sets) is called Borel σ-algebra. Sets in Borel σ-algebra are called Borel sets.
- A set which is the union of countable closed sets is called an F_σ set. A set which is the intersection of countable open sets if called a G_δ set. These sets are all Borel sets.

2 Lebesgue Measure

2.1 Outer Measure

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum l(I_n),$$

where $\{I_n\}$ is a countable collection of open intervals of lengths $l(I_n)$, and the infimum is taken over all such collections that cover A.

2.2 (Lebesgue) Measurable Sets

- E is measurable if for every set A, $m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$.
- Every subset of a set of measure zero is measurable.
- Every Borel set is measurable.
- The family of measurable sets is an algebra of sets.

2.3 Regularity of Lebesgue Measure

- Let E be a given set. The following statements are equivalent:
  (1) E is measurable.
  (2) For every $\epsilon > 0$, there is an open set $G \supset E$ such that $m^*(G - E) < \epsilon$.
  (3) For every $\epsilon > 0$, there is a closed set $F \subset E$ such that $m^*(E - F) < \epsilon$.
  (4) There is a G_δ set $G' \supset E$ such that $m^*(G' - E) = 0$.
  (5) There is a F_δ set $F' \subset E$ such that $m^*(E - F') = 0$.
- An open set of measure no more than ϵ that contains all rational numbers:
$$\bigcup_{n_k \in \mathbb{Q}, k=1,2,\ldots} (n_k - 2^{-(k+1)}\epsilon, n_k + 2^{-(k+1)}\epsilon).$$

2.4 Measurable Functions

- Let f be an extended real-valued function whose domain is measurable. Then the following statements are equivalent:
  (1) f is measurable.
  (2) The pre-image of every open set is measurable.
  (3) The pre-image of every closed set is measurable.
  (4) The pre-image of every measurable set is measurable.
- Every continuous function is measurable.
- For every sequence $\{f_n\}$ of measurable functions, the followings are all measurable: $\inf f_n, \sup f_n, \overline{\lim} f_n, \underline{\lim} f_n$.
- Let f be a measurable function defined on a set of finite measure, which takes $\pm\infty$ only on a set of measure zero. Then for every $\epsilon > 0, \delta > 0$, we can find a step function g, a continuous function h such that

  $$|f - g| < \epsilon \text{ and } f = h$$

  outside a set of measure at most δ, and a simple function p such that

  $$|f - p| < \epsilon.$$

- If $f > M$ for a.e. $x \in E$, then for every $\delta > 0$, there is $\epsilon > 0$ such that $f > M + \epsilon$ for every $x \in E$ outside a set of measure at most δ.
- If $f < \infty$ for a.e. $x \in E$, then for every $\delta > 0$, there is M such that $f \leq M$ for every $x \in E$ outside a set of measure at most δ.
- Let E be a measurable set of finite measure. $\{f_n\}$ is a sequence of measurable functions that converges to f a.e. on E. Then for every $\epsilon > 0$ and $\delta > 0$, there is a set $A \subset E$ and $m(A) < \delta$, and N such that for every $x \notin A$ and $n \geq N$,

  $$|f_n(x) - f(x)| < \epsilon.$$

3 The Lebesgue Integral

3.1 The Lebesgue Integral

- The Lebesgue integral of a simple function $\phi = \sum a_i \chi_{E_i}$, which vanishes outside a set of finite measure, is

  $$\int \phi = \int \sum a_i \chi_{E_i} = \sum a_i m(E_i)$$

- If f is a bounded measurable function on a set E of finite measure, then the Lebesgue integral of f over E is

$$\int_E f = \inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi.$$

- Let f be a nonnegative function and $\int_E f < \infty$, then for every $\epsilon > 0$ there is $\delta > 0$ such that for every set $A \in E$, if $m(A) < \delta$ then

$$\int_A f < \epsilon.$$

- If f is nonnegative and integrable, then $m(\{x : f(x) = \infty\}) = 0$.

3.2 Convergence Theorems

Let $\{f_n\}$ be a sequence of measurable functions and $\lim f_n = f$ a.e.

- (Bounded Convergence) Let E be a set of finite measure. If there is M such that $|f_n(x)| \leq M$ for all n and all $x \in E$, then

$$\int_E f = \lim \int_E f_n.$$

- (Fatou's Lemma) If f_n is nonnegative for all n, then

$$\int f \leq \underline{\lim} \int f_n.$$

- (Monotone Convergence) If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions, then

$$\int f = \lim \int f_n.$$

- (Lebesgue Convergence) If $\int_E g < \infty$ and $|f_n| < g$ on E, then

$$\int_E f = \lim \int_E f_n.$$

4 The Classical Banach Spaces

4.1 Convex Functions

- A real function ϕ is said to be convex on an interval I, if for every $x, y \in I$, $\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$.
- Examples of convex functions: x^2, e^x, $-\ln x$, x^n for $x \geq 0$, $(a + bx)^p$ for $a \geq 0$, $b > 0$, $p \geq 1$ and $x \geq 0$.

- (Jensen Inequality) Let ϕ be a convex function and f an integrable function on $[0,1]$. Then

$$\int \phi(f(t))dt \geq \phi(\int f(t)dt).$$

4.2 Linear Spaces

- A space X is called a linear space (or vector space) if for every $f, g \in X$, and any constants α, β, we have $\alpha f + \beta g \in X$.
- A norm $\|\cdot\|$ for a linear space X assigns a nonnegative real number to each $x \in X$ such that
  (1) $\|\alpha x\| = |\alpha|\|x\|$ for every constant α and $x \in X$.
  (2) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.
  (3) $\|x\| = 0$ if and only if $x = 0$.
- A normed linear space is metrizable. A norm naturally induces a metric on X such that $d(x, y) = \|x - y\|$.

4.3 The L^p Spaces

- The space $L^p = L^p[0,1]$ is defined as the space of all measurable functions on $[0,1]$ such that

$$\int_{[0,1]} |f|^p < \infty.$$

- L^p is a linear space.
- We identify $f, g \in L^p$ if $f = g$ a.e. Then for $p \geq 1$,

$$\|f\|_p = (\int_{[0,1]} |f|^p)^{\frac{1}{p}}$$

  defines a norm in L^p.
- f is called essentially bounded if there is a bounded functions g such that $f = g$ a.e. L^∞ is defined as the space of all essentially bounded functions on $[0,1]$ (functions only differ on a set of measure zero are identified). Then L^∞ is a linear space with norm

$$\begin{aligned}\|f\|_\infty = \operatorname{ess\,sup} |f(t)| &= \inf\{M : m(\{t : f(t) > M\}) = 0\} \\ &= \sup\{M : m(\{t : f(t) > M\}) > 0\}.\end{aligned}$$

- If $f \in L^p$ then $f^q \in L^{p/q}$ and $\|f^q\|_{p/q} = \|f\|_p^q$.

- (Minkowski Inequality) For $1 \le p \le \infty$ and every $f, g \in L^p$,

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

- (Hölder Inequality) If $1 \le p \le q \le \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

  and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^r$ and

$$\|f \cdot g\|_r \le \|f\|_p \cdot \|g\|_q.$$

- For $1 \le p \le q \le \infty$, $L^p \supset L^q$. For every $f \in L^q$, $\|f\|_p \le \|f\|_q$.
- An example: $f = 1/\sqrt{x}$ for a.e. $x \in [0, 1]$. $f \in L^1$ and $\|f\|_1 = 2$. But $f \notin L^2$.
- For every $1 \le p \le \infty$, $L^p \ne \bigcap_{q<p} L^q$ and $L^p \ne \bigcup_{q>p} L^q$.

4.4 Convergence and Completeness

We always assume that $p \ge 1$.

- A sequence $\{f_n\}$ in a normed linear space is said to convergence to f in the space if for every $\epsilon > 0$, there is an N such that for all $n > N$, $\|f - f_n\| < \epsilon$.
- For $1 \le p \le q \le \infty$, if $f_n \to f$ in L^q then $f_n \to f$ in L^p. The converse is false. For example,

$$f_n(x) = \begin{cases} \sqrt{n}, & 0 \le x \le 1/n, \\ 0, & 1/n < x \le 1. \end{cases}$$

  We have $f_n \to 0$ a.e., $f_n \to 0$ in L^1 but $f_n \nrightarrow 0$ in L^2.
- If $f_n \to f$ in L^p then $\|f_n\|_p \to \|f\|_p$.
- If $\|f_n\|_p \le M$ for some $M \ge 0$ for all n and $f_n \to f$ a.e. then $f_n \to f$ in L^p.
- A normed linear space is called complete if every Cauchy sequence converges. A complete normed linear space is called a Banach space.
- A normed linear space X is complete if and only if every absolutely summable series is summable.
- The L^p spaces are complete, hence are Banach spaces.

- An example: $f_{i,j} = \begin{cases} 1, & x \in [(j-1)/i, j/i] \\ 0, & \text{otherwise} \end{cases}$. The sequence

$$f_{1,1}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, f_{3,3}, f_{4,1}, \ldots$$

  converges to $f = 0$ in L^p for every p such that $1 \leq p < \infty$, while it does not converge to f almost everywhere.

4.5 Approximation in L^p

- Given $f \in L^p$, $1 \leq p \leq \infty$ and $\epsilon > 0$, there is a bounded measurable function f_M with $|f_M| \leq M$ and $\|f - f_M\| < \epsilon$. (This means that L^∞, hence L^q for $q > p$, is dense in L^p.)
- Given $f \in L^p$, $1 \leq p < \infty$ and $\epsilon > 0$, there is a step function ϕ and a continuous function ψ such that $\|f - \phi\|_p < \epsilon$ and $\|f - \psi\|_p < \epsilon$.

4.6 Bounded Linear Functionals

- A linear functional on a normed linear space X is a mapping F from X to real numbers such that $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$ for every $f, g \in X$ and real constants α, β. The linear functional F is said to be bounded if there is a constant M such that $|F(f)| \leq M\|f\|$ for every $f \in X$. The norm of F is defined as

$$\|F\| = \sup_{f \in X, f \neq 0} \frac{|F(f)|}{\|f\|}.$$

- For every $g \in L^q$, $1/p + 1/q = 1$, $F_g(f) = \int fg$ is a bounded linear functional on L^p. Moreover, $\|F_g\| = \|g\|_q$.
- Let g be an integrable function on $[0, 1]$. There is a constant M such that

$$|\int fg| \leq M\|f\|_p$$

  for every bounded measurable function f. Then $g \in L^q$ and $\|g\|_q \leq M$.
- (Riesz Representation Theorem) Let F be a bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function g in L^q such that

$$F(f) = \int fg.$$

  We also have $\|F\| = \|g\|_q$.
- The map $g \mapsto F_g$ is an isomorphism between the space of L^q functions and the space of bounded linear functionals on L^p (as normed linear spaces), where $1/p + 1/q = 1$.

- The space of bounded linear functionals on a normed linear space X is called the dual space of X. The dual space of L^p space is L^q for $1 \leq p < \infty$ and $1/p + 1/q = 1$. Note that the dual space of L^∞ is not L^1 and L^1 is not a dual space of any normed linear space.

5 Topology and Banach Spaces

5.1 Topology in L^p

- The following sets are open in L^p: $\{f : \|f\|_p < r\}$; $\{f : \int fg < r\}$.
- The following sets are closed in L^p: $\{f : \|f\|_p \leq r\}$; $\{f : \|f\|_p = r\}$, $\{f : f(x) = r, \text{ a.e. } x \in E\}$ for $m(E) > 0$.
- The set $\{f : f(x) < r, \text{ a.e. } x \in E\}$ for $m(E) > 0$ is not open in L^p.
- L^q as a subset of L^p, $p < q$, is neither open nor closed.
- L^1-norm is weaker than L^2-norm. L^1-topology is weaker than L^2-topology. i.e. If $U \subset L^2$ is open in the topology defined by L^1-norm, than U is also open in the topology defined by L^2-norm. Analogous results hold for L^p and L^q norms where $p < q$.
- C is closed in L^∞.
- For $1 \leq q \leq p$, the map $f \mapsto \|f\|_q$ is continuous in L^p.

5.2 Separability

- Definition: A space is separable if it has a countable dense subset.
- A space is not separable if it has uncountable open subsets which are pairwise disjoint.
- L^p is separable for $1 \leq p < \infty$. C is separable. L^∞ is not separable.

5.3 Compactness

- The followings are equivalent if X is a metric space.
  (1) X is compact.
  (2) Every open cover of X has a finite sub-cover (definition of compactness).
  (3) Every sequence in X has a subsequence that converges in X.
  (4) For every infinite set $E \subset X$, there is $x \in X$ such that for every open set U that contains x, there is $y \in E$, $y \neq x$ such that $y \in U$.
  (5) X is complete and totally bounded.

- A metric space X is said to be totally bounded if for every $\epsilon > 0$, X can be covered by finitely many balls of radius ϵ. For example. $(0, 1)$ is totally bounded. Any infinite space equipped with the discrete metric $d(x, y) = 1$ for $x \neq y$, is bounded but not totally bound.

- L^p is not totally bounded, hence not compact either. Sets like $\{f \in L^p : \|f\|_p \leq 1\}$ are not compact. $C[0, 1]$ is analogous (consider the family of functions $f_n(x) = (1 - 2^n x)\chi_{[0,2^{-n}]}$).

- Every continuous function maps compact sets to compact sets.

- Every continuous function on a compact metric space is uniformly continuous.

- Every closed subset of a compact set is compact.

5.4 Baire Category

- A set E in a complete metric space X is said to be nowhere dense if the complement of $\bar{E}$ is dense. This is equivalent to say that the closure of E has no interior point (contains no nonempty open set).

- If E is nowhere dense, then so is the closure of E.

- The countable union of nowhere dense sets has dense complement.

- The countable intersection of open dense sets is dense.

- A set that is a countable intersection of open dense sets is called a residual set.

- A residual set is dense and its complement is a countable union of nowhere dense sets.

- A subset of a complete metric space is residual if and only if it contains a dense G_δ set.

- Countable union or intersection of residue sets is a residual set.

- The set of all irrational numbers is a residual set in $\mathbb{R}$.

- A nowhere dense set may have arbitrary large measure. See Section 2.3. Take the complement of that set.

5.5 Equicontinuity

- A family $\mathcal{F}$ of functions from metric space X to a metric space Y is called equicontinuous at x if for any $\epsilon > 0$, there is an open set O containing x such that $d(f(x), f(y)) < \epsilon$ for all $y \in O$ and all $f \in \mathcal{F}$. $\mathcal{F}$ is called equicontinuous if it is equicontinuous at each point of X.

- (Arzelá-Ascoli) Let $\mathcal{F}$ be an equicontinuous family of functions from a separable space X to a metric space Y. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that for each $x \in X$ the closure of the set $\{f_n(x) : 0 \leq n < \infty\}$ is compact. Then there is a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function f. The convergence is uniform on each compact subset of X.

- A subset of $C[0, 1]$ is compact iff it is equicontinuous, closed and bounded.

5.6 General Topology

- d_1 is stronger than d_2 iff $id : (X, d_1) \to (X, d_2)$ is continous. d_1 and d_2 are equivalent iff $id : (X, d_1) \to (X, d_2)$ is a homeomorphism.

- d_1 is uniformly stronger than d_2 iff $id : (X, d_1) \to (X, d_2)$ is uniformly continuous. d_1 and d_2 are uniformly equivalent iff each one is uniformly stronger than the other. If X is compact, then "stronger" implies "uniformly stronger" (hence equivalent).

- If there is a constant $C > 0$ such that for $d_1(x, y) \geq Cd_2(x, y)$ for all $x, y \in X$, then d_1 is stronger than d_2.

- A topology $\mathcal{T}_1$ on X is stronger than $\mathcal{T}_2$ if $\mathcal{T}_1 \supset \mathcal{T}_2$, i.e. every open set in the space $(X, \mathcal{T}_2)$ is an open set in the space $(X, \mathcal{T}_1)$, i.e. $id : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous.

- Stronger metrics induce stronger topologies.

- Let X be a nonempty set and $\mathcal{C}$ be any collection of subsets of X. Then there is a weakest topology that contains $\mathcal{C}$.

- Let $\mathcal{F}$ be a collection of real-valued functions on X, then there is a weakest topology that contains all sets of the form $f^{-1}(U)$, where $f \in \mathcal{F}$ and U is an open set in $\mathbb{R}$. This topology is called the weak topology generated by $\mathcal{F}$. This topology is the weakest topology that makes all functions $f \in \mathcal{F}$ continuous, i.e. If $\mathcal{T}$ is a topology such that all $f \in \mathcal{F}$ is continuous, then the topology generated by $\mathcal{F}$ is weaker than $\mathcal{T}$.

- If $\mathcal{F}_1 \subset \mathcal{F}_2$, then the weak topology generated by $\mathcal{F}_1$ is weaker than the weak topology generated by $\mathcal{F}_2$.

6 General Theory of Banach Spaces

6.1 Subspace

- A nonempty subset M of a linear space X is called a subspace if $\lambda_1 x_1 + \lambda_2 x_2$ belongs to X whenever x_1 and x_2 do.

- A subspace is said to be closed if it is a closed subset.
- If M is a subspace, then so is $\overline{M}$, the closure of M.
- $C[0,1]$ is a closed subspace of $L^\infty[0,1]$.
- The subset P of all polynomials on $[0,1]$ is a subspace of $C[0,1]$, which is not closed.
- $\{f \in C[0,1] : f(0) = 0\}$ is a closed subspace of $C[0,1]$.

6.2 Linear Operators

- The norm of a bounded linear operator A between normed spaces X and Y is
$$\|A\| = \sup_{x\in X, x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|<1} \|Ax\|$$
- A linear operator maps subspaces to subspaces.
- The kernel of a linear operator $A : X \to Y$ is the set
$$\{x \in X : A(x) = 0\},$$
which is always a subspace of X. The kernel of a continuous (bounded) linear operator is a closed subspace.
- A bounded linear operator is uniformly continuous. If a linear operator is continuous at one point, it is bounded.
- The space of all bounded linear operators from a normed linear space X to a Banach space Y is itself a Banach space. The dual space of any normed linear space is a Banach space.
- Let W be a family of bounded linear operators on a Banach space X to a normed linear space Y . Then the followings are equivalent:
  (1) For every $x \in X$ there is M_x such that $|f(x)| \leq M_x$ for all $f \in W$.
  (2) There is M such that $\|f\| \leq M$ for all $f \in W$.
  (3) W is equicontinuous.

6.3 Hahn-Banach Theorem

- (Hahn-Banach) Let P be a real-valued function on a linear space X satisfying $p(x + y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$. f is a linear functional defined on a subspace S and $f(s) \leq p(s)$ for all $s \in S$. Then there is a linear functional F on X such that $F(x) \leq p(x)$ for all $x \in X$ and $F(s) = f(s)$ for $s \in S$.

- Let X be a normed linear space and $x \in X$. Then there is a bounded linear functional f such that $f(x) = \|f\|\|x\|$.
- Let T be a linear subspace of a normed linear space X and $y \in X$ such that $\|y - t\| \geq \delta \geq 0$ for all $t \in T$. Then there is a bounded linear functional f on X such that $\|f\| \leq 1$, $f(y) = \delta$ and $f(t) = 0$ for all $t \in T$.
- Let T be a linear subspace of a normed linear space X and $y \in X$. T is not dense in X. Then $\inf\{\|y - t\| : t \in T\} = \sup\{f(y) : \|f\| = 1, f(t) = 0 \text{ for all } t \in T\}$.
- There is a natural isometric isomorphism from X to its image in X^{**}: $(\phi(x))(f) = f(x), f \in X^*$. X is reflexive iff $X^{**} = \phi(X)$ iff X^* is reflexive.

6.4 The Open Mapping and Closed Graph Theorems

- A continuous linear operator from a Banach space onto another Banach space is an open mapping, i.e. it maps open sets to open sets.
- Let X be a linear space that is complete in both norms $\|\cdot\|$ and $\|\cdot\|'$. If $\|\cdot\|$ is stronger than $\|\cdot\|'$, then they are equivalent.
- Let f be a linear operator from a Banach space X to a Banach space Y, then the following are equivalent.

  (1) f is uniformly continuous.

  (2) f is bounded.

  (3) The graph of f is closed.

  (4) The kernel of f is closed.

  (5) For every $\epsilon > 0$, there is δ such that if $\|x\| < \delta$ then $\|f(x)\| < \epsilon$.

6.5 Weak Topologies

- The **weak topology** on a normed linear space X is the weak topology generated by X^*.
- A subspace of X is weakly closed if and only if it is strongly closed.
- $x_n \to x$ weakly if and only if for every $f \in X^*$, $f(x_n) \to f(x)$.
- The weak-$*$ topology on X^* is the weak topology generated by X (indeed, by $\phi(X) \subset X^{**}$).
- If X is reflexive, then the weak topology on X^* is the same as the weak-$*$ topology.
- The unit sphere $S = \{f \in X^* : \|f\| = 1\}$ is compact in the weak-$*$ topology. So is $S_r = rS = \{f \in X^* : \|f\| \leq r\}$ for every $r > 0$.

- Let W be a bounded subset of a Banach space X, then
  (1) The weak-$*$ closure of W is compact in the weak-$*$ topology.
  (2) The closed convex hull of W is compact in the weak-$*$ topology.
  (3) For every sequence in W, there is a subsequence $\{f_{k_n}\}$ and $f \in X^*$ such that for every $x \in X$, $f_{k_n}(x)$ converges to $f(x)$.
- The norm topology as well as the weak topology on a normed linear space makes the space a topological vector space, such that:
  (1) The maps $f(x) = \alpha x$ for $\alpha \in \mathbb{R}$ and $g(x) = x + a$ for $a \in X$ are continuous.
  (2) The topological properties (such like open, closed, compact, dense, and so on) are the same for a set A, αA for $\alpha > 0$ and $A + a$ for $a \in X$.

6.6 Convexity

- A set K in a linear space X is called convex if $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $\lambda \in [0, 1]$.
- If K_1, K_2 are convex then $aK_1 + bK_2$ is convex for all $a, b \in \mathbb{R}$, $K_1 \cap K_2$ is also convex, while $K_1 \cup K_2$ may not be convex.
- A subspace is always convex.
- Every normed linear space is **locally convex** in the norm topology as well as in weak topology generated by any collection of bouned linear functionals.
- Every convex closed set in a locally convex space is weakly closed.
- Every bounded, (strongly) closed, convex subset of L^p, $1 < p < \infty$, is weakly compact.
- Let K be a convex set. $x \in K$ is called an extreme point if there are not two elements $y, z \in K$ such that $y + z = 2x$.
- The convex hull of E is the intersection of all convex sets containing E. The closed convex hull of E is the intersection of all closed convex sets containing E.
- Let K be a compact convex set in a locally convex space. Then K is the closed convex hull of its extreme points.
- $L^1[0, 1]$ and $C[0, 1]$ are not dual spaces for any normed linear spaces.

6.7 Hilbert Spaces

- A Hilbert space is a Banach space with an inner product such that
  1. $(ax + by, z) = a(x, z) + b(y, z)$.
  2. $(x, y) = (y, x)$.
  3. $(x, x) = \|x\|^2$.
- $\mathbb{R}^n$ is a Hilbert space with $(x, y) = \sum_{i=1}^{n} x_i y_i$.
- L^2 is a Hilbert space with $(f, g) = \int fg$.
- For every x, y in a Hilbert space, $(x, y) \leq \|x\| \cdot \|y\|$.
- x, y in a Hilbert space is called orthogonal if $(x, y) = 0$. Subspaces M and N are called orthogonal if $(x, y) = 0$ for all $x \in M$ and $y \in N$.
- Let f be a bounded linear functional on the Hilbert space H. Then there is a unique $y \in H$ such that $f(x) = (x, y)$ for all x. Moreover, $\|f\| = \|y\|$.