DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **27**, Number **3**, July **2010**

doi:10.3934/dcds.2010.27.1219

pp. **1219–1231**

MEASURES OF INTERMEDIATE ENTROPIES FOR SKEW PRODUCT DIFFEOMORPHISMS

Peng Sun

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA

(Communicated by Yuri Kifer)

Abstract. In this paper we study a skew product map F preserving an ergodic measure μ of positive entropy. We show that if on the fibers the map are $C^{1+\alpha}$ diffeomorphisms with nonzero Lyapunov exponents, then F has ergodic measures of arbitrary intermediate entropies. To construct these measures we find a set on which the return map is a skew product with horseshoes along fibers. We can control the average return time and show the maximal entropy of these measures can be arbitrarily close to $h_\mu(F)$.

1. **Introduction.** Entropy has been one of the centerpieces in dynamics. It reflects the complexity of the system. For smooth systems, positive topological entropy comes from some (partially) hyperbolic structure, and is conjectured to be accompanied with plenty of invariant measures. The work presented here explores answers to such conjectures in a case of skew product maps.

We consider a compact Riemannian manifold M and a $C^{1+\alpha}$ ($\alpha > 0$, this means the derivative is α-Hölder continuous) diffeomorphism f on M. If f preserves an ergodic measure μ, then there are real numbers λ_k, $k = 1, 2, \ldots, l$, such that for μ-almost every point x, there are subspaces $E_k(x)$ of the tangent space T_xM such that for any vector $v \in E_k(x)\backslash\{0\}$,

$$\lambda(v) = \lim_{n\to\infty} \frac{\log \|df^n v\|}{n} = \lambda_k$$

Here l is some integer number no more than the dimension of M and these subspaces are invariant of df, the derivative of f. These numbers λ_k are called Lyapunov exponents. We say μ is a hyperbolic measure if all Lyapunov exponents are nonzero. Three decades ago, A. Katok established a well-known result as following:

Theorem 1.1. (Katok, [6, 7]) *If the metric entropy $h_\mu(f) > 0$ and μ is an ergodic hyperbolic measure, then for any $\epsilon > 0$, there is a hyperbolic horseshoe $\Lambda \subset M$ such that $h(f|_\Lambda) > h_\mu(f) - \epsilon$.*

This theorem has an interesting corollary: Under the conditions of the theorem, there are ergodic measures μ_β such that $h_{\mu_\beta}(f) = \beta$ for any real number $\beta \in [0, h_\mu(f)]$, i.e. all possible entropies of ergodic measures form an interval. Since the

---

2000 *Mathematics Subject Classification.* Primary: 37D25, 37C40; Secondary: 37A05.

Key words and phrases. entropy, ergodic measure, Lyapunov exponents, skew product, horseshoe, return map, Pesin theory.

horseshoe map is a full shift, these measures can be constructed by taking sub-shifts or properly assigning weights to different symbols. Existence of these measures of intermediate entropies exhibits the complicated structure of the system.

Till now it is not known whether every smooth system ($C^{1+\alpha}$ diffeomorphism) on a compact manifold with positive (topological) entropy has this property. In general, such a system may not have a hyperbolic measure. Herman constructed a well-known example, which is a minimal C^∞ diffeomorphism with positive topological entropy [4]. So in the general case even a closed invariant subset should not be expected. Fortunately, minimality may not prevent the system from having measures of intermediate entropies, which is the case in Herman's example. Herman suggested the question whether for every smooth system positive topological entropy violates unique ergodicity. Katok conjectured more ambitiously: these systems must have ergodic invariant measures of arbitrary intermediate entropies. By variational principle, this holds if given any ergodic measure with positive entropy $h_\mu(f)$ there are measures of entropies between zero and $h_\mu(f)$.

In this paper we deal with skew product maps with nonzero Lyapunov exponents along fibers. Let $F = (g, f_x)$ be a skew product map on the space $X \times Y$ preserving an ergodic measure $\mu = \int \sigma_x d\nu$. We assume that g is an invertible (mod 0) measure preserving transformation on the probability space (X, ν). Then g is ergodic. For every $x \in X$, f_x is a $C^{1+\alpha}$ diffeomorphism on the compact Riemannian manifold Y, sending σ_x to $\sigma_{g(x)}$.

Main Theorem. *Assume that $h_\mu(F) > 0$ and $h_\nu(g) = 0$. If for almost every $z = (x, y) \in X \times Y$ and every $v \in T_y(\{x\} \times Y)\backslash\{0\}$, the Lyapunov exponent*

$$\lambda(v) = \lim_{n\to\infty} \frac{\log \|df_{g^{n-1}(x)} \cdots df_{g(x)} df_x v\|}{n} \neq 0,$$

then F has ergodic invariant measures of arbitrary intermediate entropies.

Remark 1. Our proof also works when $h_\nu(g) > 0$. In this case it concludes that F has ergodic measures of arbitrary entropies between $h_\nu(g)$ and $h_\mu(F)$.

In addition, we assume g has no periodic point. Otherwise the problem is reduced to Theorem 1.1. We have shown in [9] that under the conditions there are measures of zero entropy. Some lemmas are generalized and adapted in this paper.

2. **Entropy and separated sets on fibers.**

In this section we discuss the entropy of the skew product and obtain an estimate on the cardinality of the (m, ϵ)-separated set on each fiber, which is analogous to the definition of metric entropy by Katok [5].

Let η be a measurable partition of the fiber Y such that

$$\int_X H_x(\eta) d\nu < \infty \tag{1}$$

where $H_x(\eta) = -\sum_{\mathcal{C}\in\eta} \sigma_x(\mathcal{C}) \log \sigma_x(\mathcal{C})$. We put

$$\eta_x^n = \bigvee_{k=1}^{n} f_x^{-1} f_{g(x)}^{-1} \cdots f_{g^{k-1}(x)}^{-1} \eta$$

Theorem 2.1. (Abramov and Rohlin, [1]) *For every η satisfying (1), let*

$$h^g(f, \eta) = \lim_{n\to\infty} \frac{1}{n} \int_X H_x(\eta_x^n) d\nu$$

The limit exists and it is finite. Let

$$h^g(f) = \sup_\eta h^g(f, \eta)$$

$h^g(f)$ *is called the fiber entropy. We have*

$$h_\mu(F) = h_\nu(g) + h^g(f)$$

For the skew product map we consider, g is ergodic and, for almost every $x \in X$, $\sigma_x \circ f_x^{-1} = \sigma_{g(x)}$. we have for almost every x,

$$h_x^g(f, \eta) = \lim_{n\to\infty} \frac{1}{n} H(\eta_x^n) = h^g(f, \eta)$$

The following is a version of Shannon-McMillan-Breiman Theorem for skew product maps.

Theorem 2.2. (Belinskaja, [3]) *Let $\mathcal{C}_x^n(y)$ be the element of η_x^n containing y. For almost every $(x, y) \in X \times Y$,*

$$\lim_{n\to\infty} -\frac{1}{n} \log \sigma_x(\mathcal{C}_x^n(y)) = h_x^g(f, \eta) = h^g(f, \eta)$$

Fix d a Riemannian metric on Y. Let $d_n^F(z, z')$ be the metrics defined for z, z' on the same fiber $\{x\} \times Y$ by:

$$d_n^F(z, z') = \max_{0\le i\le n-1} d(F^i z, F^i z')$$

For $x \in X$ and $\delta > 0$, on the fiber $\{x\} \times Y$, let $\mathcal{N}_x^F(n, \epsilon, \delta)$ be the minimal number of ϵ-balls in the d_n^F-metric needed to cover a set of σ_x-measure at least $1 - \delta$, and let $\mathcal{S}_x^F(n, \epsilon, \delta)$ be the maximal number of (d_n^F, ϵ)-separated points we can find inside every set of σ_x-measure at least $1 - \delta$.

We can follow exactly Katok's argument for [5, Theorem 1.1] and obtain the analogous result:

Theorem 2.3. *If F is ergodic, then for every $\delta \in (0, 1)$ and almost every $x \in X$,*

$$\begin{aligned} h^g(f) &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{N}_x^F(n, \epsilon, \delta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{N}_x^F(n, \epsilon, \delta)}{n} \\ &= \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \mathcal{S}_x^F(n, \epsilon, \delta)}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log \mathcal{S}_x^F(n, \epsilon, \delta)}{n} \end{aligned}$$

3. **Recurrence.** In this section we discuss some properties related to nontrivial recurrence of the map.

Theorem 3.1 is a generalization of [9, Proposition 3.2]. It says that, for a subset of positive measure, if the conditional measures are uniformly bounded from below, then on each fiber we can find points that return relatively faster, such that the return time is integrable. In this paper we still use only the special version for the first return.

Theorem 3.2 is crucial in the proof of the main theorem. We consider a complicated or even randomly selected return map on some set of positive measure, which may be neither injective nor surjective. We show if the return time is integrable, then this return map is one-to-one and measure preserving, on a smaller subset of positive measure. We call this subset the *kernel* of the return map. Moreover, we have an estimate of the integral of the return time, which may be used to estimate the size (measure) of the kernel.

Theorem 3.1. (Integrability of Return Time) *Let $P \subset X \times Y$ be a measurable subset. $B = \pi(P) \subset X$ is the projection of P on the base. For $x \in B$, denote $P \cap (\{x\} \times Y)$ by $P(x)$. Assume that $\nu(B) = \nu_0 > 0$ and there is $\sigma_0 > 0$ such that for (almost) every $x \in B$, $\sigma_x(P(x)) > \sigma_0$. Hence $\mu(P) = \mu_0 > \nu_0\sigma_0 > 0$. For (almost) every $z \in P$, denote by $n_l(z)$ the l-th return time of z. Let*

$$P_n^l(x) = \{z \in P(x)|\ n_l(z) \geq n\},\ N_l(x) = \max\{n|\sigma_x(P_n^l(x)) > \sigma_0\}$$

Then

$$\int_B N_l(x)d\nu < \frac{l}{\sigma_0} < \infty$$

Remark 2. $N_l(x)$ is the longest return time for the l-th returns of the points in a subset of conditional measure no less than $\sigma_x(P(x)) - \sigma_0$ in $P(x)$. Or equivalently, $N_l(x)$ is the smallest number such that the set of points in $P(x)$ with l-th return times greater than $N_l(x)$ has conditional measure at most σ_0.

Proof. Since μ is F-invariant and $\mu(P) > 0$, we have

$$0 < \int_P n_1(z)d\mu = \mu(\bigcup_{j=0}^{\infty} F^j(P)) \leq 1$$

Let F_P be the first return map on P, which preserves μ, then for each k,

$$\int_P n_{k+1}(z)d\mu = \int_P (n_k(F_P(z)) + n_1(z))d\mu = \int_P n_k(z)d\mu + \int_P n_1(z)d\mu$$

Hence

$$0 < \int_P n_l(z)d\mu = l\int_P n_1(z)d\mu \leq l$$

Note

$$\int_P n_l(z)d\mu = \sum_{j=1}^{\infty} \mu(P_j)$$

where $P_j = \{z \in P|n_l(z) \geq j\}$ consists of points with l-th return time no less than j. Note $P_j^l(x) = P_j \cap P(x)$.

For every $x \in B$, let $B^j = \{x \in B|N_l(x) \geq j\}$. By definition, for every $j \leq N_l(x)$, $\sigma_x(P_j^l(x)) > \sigma_0$. So $x \in B^j$ iff $\sigma_x(P_j^l(x)) = \sigma_x(P_j \cap P(x)) > \sigma_0$. We have

$$\mu(P_j) = \int_B \sigma_x(P_j \cap P(x))d\nu > \int_{B^j} \sigma_0 d\nu = \nu(B^j) \cdot \sigma_0$$

hence

$$\int_B N_l(x)d\nu = \sum_{j=1}^{\infty} \nu(B^j) < \sum_{j=1}^{\infty} \frac{1}{\sigma_0}\mu(P_j) \leq \frac{l}{\sigma_0}$$

□

Theorem 3.2. *Let g be a measure preserving transformation on a probability space (X, ν). g is invertible and has no periodic point. B is a subset of X with $\nu(B) = \nu_0 > 0$. $N : B \to \mathbb{N}$ is a measurable function such that $\tilde{g}(x) = g^{N(x)}(x) \in B$ for almost every $x \in B$. Assume*

$$\int_B N(x)d\nu = \Sigma_B < \infty \tag{2}$$

Then there is a subset B' of B such that the following holds:

1. *$\nu(B') = \nu_1 > 0$, $\tilde{g}(B') = B'$ and $g_* = \tilde{g}|_{B'}$ is invertible and ν-preserving.*

2.
$$\int_{B'} N(x)d\nu \geq \nu(\bigcup_{j=-\infty}^{\infty} g^j(B))$$

We call B' the kernel of $\tilde{g}$.

Remark 3. The assumption that g has no periodic point is not necessary in this theorem. Periodic orbits may be removed as a null set or we can easily find a subset consisting of periodic orbits on which $\tilde{g}$ is invertible.

Remark 4. A result similar to the first part of the theorem has been discussed in [8] (thanks to the referee), where B coincides with X and the measurable function N takes values in $\mathbb{Z}$. Our proof is obtained independently with a few different tricks.

Proof. With possible loss of a null set we may assume that the first return map g_B on B is defined everywhere on B and invertible. g_B has no periodic point since g has not.

Define a partial order on B: $x_1 \prec x_2$ iff there is $n \geq 0$ such that $g_B^n(x_1) = x_2$, i.e. x_2 is an image of x_1 under iterates of g_B (and g). Since g_B is invertible and has no periodic point, this partial order is well defined.

Let $O^+(x) = \{\tilde{g}^k(x) | k \in \mathbb{N} \cup \{0\}\}$ be the forward $\tilde{g}$-orbit of x. We define an equivalence relation on B: $x_1 \sim x_2$ iff $O^+(x_1) \cap O^+(x_2) \neq \emptyset$, i.e. there are $k_1, k_2 > 0$ such that $\tilde{g}^{k_1}(x_1) = \tilde{g}^{k_2}(x_2)$. Note $x_1 \in O^+(x_2)$ or $x_2 \in O^+(x_1)$ implies $x_1 \sim x_2$, but the converse is not true. Also note within an equivalence class the partial order is a total order, since g_B is invertible and $\tilde{g}^{k_1}(x_1) = \tilde{g}^{k_2}(x_2) = x_3$ implies $g_B^{n_1}(x_1) = g_B^{n_2}(x_2)$ for some n_1 and n_2. We write $x_1 \precsim x_2$ if $x_1 \prec x_2$ and $x_1 \sim x_2$.

Here we adapt some facts we showed in [9]. The integrability condition (2) is necessary for the following two lemmas.

Lemma 3.3. ([9], Proposition 4.1) *For almost every $x \in B$, there is $x^* \in B$ such that for every $x' \precsim x$, $x^* \in O^+(x')$, i.e.*
$$H(x) = \bigcap_{x' \precsim x} O^+(x') \neq \emptyset$$
Moreover, if $x_1 \precsim x_2$ then $H(x_1) \supset H(x_2)$.

Lemma 3.4. ([9], Proposition 4.2) *For almost every $x \in B$, there is a point x' such that $x' \precsim x$ and $x' \neq x$. Hence for almost every $x \in B$, there are infinitely many x' such that $x' \precsim x$.*

Excluding a null set and its (full) g-orbit (the union is still a null set), we can assume the results in the last two lemmas hold for every $x \in B$.

Let $\tilde{B} = \bigcap_{j=1}^{\infty} \tilde{g}^j(B)$. Then $\tilde{B}$ is a measurable subset of B, consisting of elements that lie in the forward $\tilde{g}$-orbits of infinitely many elements of B. For every $x \in B$, let
$$G(x) = \bigcup_{x' \sim x} H(x')$$
Then for every $\tilde{x} \in G(x)$, there is x' such that $x' \sim x$ and
$$\tilde{x} \in H(x') = \bigcap_{x'' \precsim x'} O^+(x'')$$
By Lemma 3.4, the intersection is of infinitely many forward $\tilde{g}$-orbits, which implies $\tilde{x} \in \tilde{B}$. So $G(x) \subset \tilde{B}$ for every $x \in B$.

For $x \in \tilde{B}$, there are infinitely many elements in B such that their forward $\tilde{g}$-orbits pass through x. They also pass through one of the elements in the pre-image $\tilde{g}^{-1}(x)$. But by integrability of return time (2), the pre-image $\tilde{g}^{-1}(x)$ consists of finite number of elements. By pigeonhole principle, there must be some element in $\tilde{g}^{-1}(x)$ which lies in infinitely many forward $\tilde{g}$-orbits. Such an element belongs to $\tilde{B}$, hence $\tilde{g}^{-1}(x) \cap \tilde{B}$ is nonempty. The function

$$N'(x) = \min\{N(\tilde{x}) | \tilde{g}(\tilde{x}) = x, \tilde{x} \in \tilde{B}\}$$

is a well-defined measurable function on $\tilde{B}$.

Define $g'(x) = g^{-N'(x)}(x)$. For $x \in \tilde{B}$ note $g'(x) \in \tilde{B}$ and hence $g'(\tilde{B}) \subset \tilde{B}$. So g' is a measurable transformation on $\tilde{B}$. We also note that $\tilde{g}(g'(x)) = x$ for every $x \in \tilde{B}$.

Let $B' = \bigcap_{j=1}^{\infty} (g')^j(\tilde{B})$. Then B' is measurable. On one hand, by definition we have $g'(B') = B'$. For every $x \in B'$, $g'(x) \in B'$ and $\tilde{g}(g'(x)) = x$. This implies $\tilde{g}(B') \supset B'$. On the other hand,

$$\tilde{g}(B') = \tilde{g}(\bigcap_{j=1}^{\infty} (g')^j(\tilde{B})) \subset \bigcap_{j=1}^{\infty} \tilde{g}((g')^j(\tilde{B})) = \bigcap_{j=1}^{\infty} (g')^{j-1}(\tilde{B}) = B'$$

So $\tilde{g}(B') = B'$.

For every $x \in B$ and every $\tilde{x} \in G(x)$, $\tilde{g}(\tilde{x}) \in G(x) \subset \tilde{B}$. We claim

Lemma 3.5. *If $\tilde{x} \in G(x)$, then $g'(\tilde{g}(\tilde{x})) = \tilde{x}$.*

Proof. Assume $g'(\tilde{g}(\tilde{x})) = x_0 \neq \tilde{x}$. Since $\tilde{x} \in G(x)$, there is x' such that $\tilde{x}$ lies on the forward $\tilde{g}$-orbit of every x'' such that $x'' \precsim x'$, while $x_0 \in \tilde{B}$ lies in infinitely many forward $\tilde{g}$-orbits of the elements in the same equivalence class with x'. As g is invertible, there are only finitely many elements between (in the sense of the partial order) x' and x_0. So there must be some (in fact, infinitely many) x_1 such that $x_1 \precsim x'$ and both $\tilde{x}$ and x_0 lie on the forward $\tilde{g}$-orbit of x_1. Assume $\tilde{x} = \tilde{g}^a(x_1)$ and $x_0 = \tilde{g}^b(x_1)$. Then $\tilde{g}^{a+1}(x_1) = \tilde{g}^{b+1}(x_1) = \tilde{g}(\tilde{x})$. As g has no periodic orbit and $\tilde{g}$ as well, we must have $a = b$ and $\tilde{x} = x_0$, which is a contradiction. □

From the lemma we know for every $\tilde{x} \in G(x)$, $(g')^k(\tilde{g}^k(\tilde{x})) = \tilde{x}$, hence $\tilde{x} \in (g')^k(\tilde{B})$ for every positive integer k. This yields

Corollary 1. *$G(x) \subset B'$ for every $x \in B$.*

Furthermore,

$$\bigcup_{j=-\infty}^{\infty} g^j(B') \supset \bigcup_{j=-\infty}^{\infty} g^j(\bigcup_{x \in B} G(x)) \supset B \tag{3}$$

In particular, B' is nonempty and has positive measure. We shall show $\tilde{g}$ is invertible on B'.

$\tilde{g}|_{B'}$ is surjective since we have showed $\tilde{g}(B') = B'$.

$\tilde{g}|_{B'}$ is injective. If $x \in B'$ then $x \in g'(\tilde{B})$ and there is $\tilde{x} \in \tilde{B}$ such that $x = g'(\tilde{x})$. But $\tilde{x} = \tilde{g}(g'(\tilde{x})) = \tilde{g}(x) \in B'$. This implies that $x = g'(\tilde{g}(x))$ for every $x \in B'$. So if $x_1, x_2 \in B'$ and $\tilde{g}(x_1) = \tilde{g}(x_2) \in B'$, then $x_1 = g'(\tilde{g}(x_1)) = g'(\tilde{g}(x_2)) = x_2$.

$g_* = \tilde{g}|_{B'}$ preserves ν. Let $D_k = \{x \in B' | N(x) = k\}$ for $k = 1, 2, \cdots$. Then $B' = \bigcup_{k=1}^{\infty} D_k$ and $D_i \bigcap D_j = \emptyset$. $g_*(D_i) \bigcap g_*(D_j) = \emptyset$ for $i \neq j$ since g_* is invertible.

$g_*|_{D_k} = g^k$ preserves ν. For any measurable subset $E \subset B'$, $E = \bigcup_{1\leq n<\infty} E_n$, where $E_n = E \bigcap D_n \subset D_n$. We have

$$\nu(g_*(E)) = \sum_{1\leq n<\infty} \nu(g_*(E_n)) = \sum_{1\leq n<\infty} \nu(E_n) = \nu(E)$$

This completes the proof of the first part.

For the second part, consider

$$B'' = \bigcup_{k=1}^{\infty}(\bigcup_{j=0}^{k-1} g^j(D_k)) = \bigcup_{j=0}^{\infty} g^j(\bigcup_{k=j+1}^{\infty} D_k) \subset \bigcup_{j=0}^{\infty} g^j(B') \tag{4}$$

Lemma 3.6. *B'' is g-invariant, and*

$$B'' = \bigcup_{j=-\infty}^{\infty} g^j(B)$$

Proof. First note for each k, $g^k(D_k) = \tilde{g}(D_k) \subset B' \subset B''$. Then

$$\bigcup_{k=1}^{\infty} g^k(D_k) = \bigcup_{k=1}^{\infty} \tilde{g}(D_k) = \tilde{g}(B') = B' = \bigcup_{k=1}^{\infty} g^0(D_k)$$

So

$$g(B'') = \bigcup_{k=1}^{\infty}(\bigcup_{j=0}^{k-1} g^{j+1}(D_k)) = (\bigcup_{k=1}^{\infty}(\bigcup_{j=1}^{k-1} g^j(D_k))) \cup (\bigcup_{k=1}^{\infty} g^k(D_k)) = B''$$

From (4) and $B' \subset B''$, we have

$$\bigcup_{j=-\infty}^{\infty} g^j(B') \subset \bigcup_{j=-\infty}^{\infty} g^j(B'') = B'' \subset \bigcup_{j=0}^{\infty} g^j(B')$$

which implies

$$\bigcup_{j=-\infty}^{\infty} g^j(B') = B''$$

From (3),

$$B \subset \bigcup_{j=-\infty}^{\infty} g^j(B') \subset \bigcup_{j=-\infty}^{\infty} g^j(B)$$

So

$$\bigcup_{j=-\infty}^{\infty} g^j(B) = \bigcup_{j=-\infty}^{\infty} g^j(B') = B''$$

∎

Lemma 3.6 yields

$$\int_{B'} N(x)d\nu = \sum_{k=1}^{\infty} k\cdot\nu(D_k) \geq \nu(B'') = \nu(\bigcup_{j=-\infty}^{\infty} g^j(B))$$

∎

Corollary 2. *Let C be a subset of B' of positive measure such that $g_*(C) = C$. If g is ergodic, then*

$$\int_C N(x)d\nu \geq 1$$

Proof. Let $C_k = \{x \in C | N(x) = k\}$ for $k = 1, 2, \cdots$. Consider

$$C' = \bigcup_{k=1}^{\infty} (\bigcup_{j=0}^{k-1} g^j(C_k))$$

Similar argument shows $g(C') = C'$. If g is ergodic, then $\nu(C') = 1$. Hence

$$\int_C N(x) d\nu = \sum_{k=1}^{\infty} k \cdot \nu(C_k) \geq \nu(C') = 1$$

□

4. **Proof of the Main Theorem.**

4.1. **Regular tube.** Assume on the fiber direction there is no zero Lyapunov exponents. If $h^g(f) > 0$, then there must be at least one positive exponent and one negative exponent. From Pesin theory [2] we know for almost every point z there is a regular neighborhood around z on the fiber. Inside each regular neighborhood we can introduce a local chart and identify a "rectangle" with the square $[-1, 1]^2$ (z with 0) in Euclidean space (in higher dimension this should be recognized as the product of unit balls in dimensions corresponding to contracting and expanding directions).

Fix some small number $\gamma > 0$, we can define admissible (s, γ)-curves as the graphs $\{(\theta, \psi(\theta)) | \theta \in [-1, 1]\}$ and admissible (u, γ)-curves as $\{(\psi(\theta), \theta) | \theta \in [-1, 1]\}$, if $\psi : [-1, 1] \to [-1, 1]$ is a C^1 map with $|\psi'| < \gamma$. There is some $0 < h < 1$ such that, if in addition $|\psi(0)| < h$ then admissible (s, γ)-curves are mapped by F^{-1}, while (u, γ)-curves by F, to the admissible curves of the same types, respectively.

Consider admissible (s, γ)-rectangles defined as the sets of points

$$\{(u, v) \in [-1, -1]^2 | v = \omega\psi_1(u) + (1 - \omega)\psi_2(u), 0 \leq \omega \leq 1\}$$

if ψ_1 and ψ_2 are admissible (s, γ)-curves. Admissible (u, γ)-rectangles are defined analogously. Like admissible curves, these admissible rectangles are also mapped to admissible rectangles of the same types by F^{-1} and F, respectively.

Let us fix small numbers $\epsilon > 0$ and $r > 0$.

Proposition 1. *There is a "Regular Tube" P, which is a measurable subset of $X \times Y$ satisfying the following properties:*

1. $\mu(P) = \mu_0 > 0$.
2. *Let $\pi : P \to X$ be the projection to the base and let $B = \pi(P)$. Then $\nu(B) = \nu_0 > 0$.*
3. *Let $P(x) = P \cap (\{x\} \times Y)$. There is some number $\sigma_0 > 0$ such that, for every $x \in B$, $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1 + r)$.*
4. *For every $x \in B$, there is a rectangle $R(x)$ on the fiber $\{x\} \times Y$ whose diameter is less than $\epsilon/2$. $P(x) \subset R(x) \subset \mathcal{R}(z)$ where $\mathcal{R}(z)$ is the Lyapunov regular neighborhood of some point $z = (x, y) \in X \times Y$ on the fiber $\{x\} \times Y$.*
5. *For every $x \in B$ and $z \in P(x)$, if for some $n > 0$, $F^n(z)$ returns to P, i.e. $g^n(x) \in B$ and $F^n(z) \in P(g^n(x))$, then the connected component of the intersection $F^n(R(x)) \cap R(g^n(x))$ containing $F^n(z)$, denoted by*

   $$CC(F^n(R(x)) \cap R(g^n(x)), F^n(z))$$

   is an admissible (u, γ)-rectangle in $R(g^n(x))$ and

   $$CC(F^{-n}(R(g^n(x))) \cap R(x), z)$$

is an admissible (s,γ)*-rectangle in* $R(x)$*. Moreover, for* $j = 0, 1, ..., n$*, on the fiber* $\{g^j(x)\} \times Y$ *we have*

$$\mathrm{diam} F^j(CC(F^{-n}(R(g^n(x))) \cap R(x), z)) < \epsilon$$

6. *Applying Theorem 2.3, we may assume that there is some* $m_1 > 0$ *such that for every* $m > m_1$ *and every* $x \in B$*, inside any set of* σ_x*-measure at least* $\sigma_0/2$ *on the fiber* $\{x\} \times Y$*, we can find a* (d_m^F, ϵ)*-separated set with cardinality at least* $\exp m(h_\mu(F) - r)$*.*

Proof. This regular tube can be obtained with the following steps.

1. On almost every fiber, find a regular point $z \in \{x\} \times Y$ and its regular neighborhood $\mathcal{R}(z)$. Take $R(x) \in \mathcal{R}(z)$ with diameter less than $\epsilon/2$.
2. Find m_1 and $B_0 \in X$ with $\nu(B_0) > 0$ such that property (6) holds for $m > m_1$ and $x \in B_0$.
3. Find $P(x) \subset R(x)$ satisfying property (5). There is some $\sigma_0 > 0$ such that $B = \{x \in B_0 | \sigma_x(P(x)) > \sigma_0\} > 0$. For $x \in B$, shrink the size of $P(x)$ properly such that $\sigma_0 < \sigma_x(P(x)) < \sigma_0(1 + r)$. $P = \bigcup_{x\in B} P(x)$ is as required.

□

4.2. **Control of return time.** We start with a regular tube P. Applying Theorem 3.1, we can find a measurable section $q : B \to P$, $\pi \circ q = Id$ such that

$$\int_B N_1(x) d\nu \leq \frac{1}{\sigma_0}$$

where $N_1(x)$ is the first return time of $q(x)$.

Denote by χ_P the characteristic function of the measurable set P. Consider the sets

$$\mathcal{A}_n = \{z \in X \times Y | \text{ For every } k \geq n, \sum_{i=1}^{k} \chi_P(F^i z) < k\mu_0(1 + \frac{r}{3}) \\ \text{and } \sum_{i=1}^{k(1+r)} \chi_P(F^i z) > k\mu_0(1 + \frac{2r}{3})\}$$

(throughout this paper, numbers like $k(1 + r)$ are rounded to the nearest integer, if needed). Since μ is ergodic, Birkhoff Theorem tells us for almost every z,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_P(F^i z) = \mu(P) = \mu_0$$

which implies

$$\lim_{n\to\infty} \mu(\mathcal{A}_n) = 1$$

Let $\mathcal{B}_n = \{x | \sigma_x(P(x) \cap \mathcal{A}_n) > \sigma_0(1 - r)\}$. Then as $n \to \infty$,

$$\nu(B \backslash \mathcal{B}_n) \to 0 \text{ and } \int_{B\backslash\mathcal{B}_n} N_1(x) d\nu \to 0$$

There is m_0 and a measurable subset $B_1 \subset B$ with the following properties

1. For $x \in B_1$, let $P^{(m)}(x) = P(x) \cap \mathcal{A}_m$. For $m > m_0$, $\sigma_x(P^{(m)}(x)) > \sigma_0(1 - r)$.
2. $\nu(B_1) > \nu_0(1 - r)$. Let $B_2 = B \backslash B_1$. Then

$$\int_{B_2} N_1(x) d\nu < r$$

Now we fix $m > \max\{m_1, m_0\}$. For convenience, denote by $\mathcal{K} = [m\mu_0(1+\frac{r}{2})]$ the integer part of $m\mu_0(1+\frac{r}{2})$. For large m,

$$m\mu_0(1+\frac{r}{3}) < \mathcal{K} < m\mu_0(1+\frac{2r}{3})$$

For every $x \in B_1$, $P^{(m)}(x) > \sigma_0(1-r) > \sigma_0/2$, by property (6) of the regular tube, there is a (d_m^F, ϵ)-separated set $E(x) \subset P^{(m)}(x)$ with cardinality

$$|E(x)| > \exp m(h_\mu(F) - r)$$

For $z \in E(x) \subset \mathcal{A}_m$, the $\mathcal{K}$-th return time of z to P is an integer number between $m+1$ and $m(1+r)$. So there is $V(x) \subset E(x)$ with cardinality

$$|V(x)| = [\frac{1}{mr} \exp m(h_\mu(F) - r)]$$

and the $\mathcal{K}$-th return times for points in $V(x)$ are the same, denoted by $N(x)$. The set $\bigcup_{x\in B_1} V(x)$ can be chosen to be the union of $[\frac{1}{mr}\exp m(h_\mu(F)-r)]$ measurable sections over B_1.

$N(x)$ is a measurable function on B_1. We extend $N(x)$ to a measurable function on B: for $x \in B_2$, let $N(x) = N_1(x)$. Consider the map $\tilde{g}(x) = g^{N(x)}(x)$. $\tilde{g}(x)$ is well defined on B and $\tilde{g}(B) \subset B$. Moreover,

$$\int_B N(x)d\nu \le \int_{B_1} N(x)d\nu + \int_{B_2} N_1(x)d\nu < m(1+r)\cdot\nu_0 + r < \infty$$

Applying Theorem 3.2 we can find the kernel $B_3 \subset B$ of positive measure such that the $\tilde{g}$ restricted on B_3 is invertible and preserves μ. We can assume that $g_* = \tilde{g}|_{B_3}$ is ergodic (with respect to the measure induced by ν) by taking an ergodic component of positive ν-measure.

Let $B_4 = B_3 \cap B_1$ and $B_5 = B_3 \cap B_2$. Let $\mathcal{G}(x)$ be the first return map on B_4 with respect to g_*. Then $\mathcal{G}$ is invertible and preserves ν. Define measurable functions ρ and l on B_4 such that $\mathcal{G}(x) = g^{\rho(x)}(x) = g_*^{l(x)}(x)$. For $x \in B_4$,

$$\rho(x) = \sum_{j=0}^{l(x)-1} N(g_*^j(x))$$

and

$$\int_{B_4} \rho(x)d\nu = \int_{B_3} N(x)d\nu$$

Note $B_3 = B_4 \cup B_5 \subset B_4 \cup B_2$. From Corollary 2,

$$1 \le \int_{B_3} N(x)d\nu \le \int_{B_4} N(x)d\nu + \int_{B_2} N(x)d\nu \le m(1+r)\cdot\nu(B_4) + r$$

So

$$\nu(B_4) \ge \frac{1-r}{m(1+r)}$$

and the average return time

$$\begin{aligned}\frac{1}{\nu(B_4)}\int_{B_4}\rho(x)d\nu \le& \frac{m(1+r)\cdot\nu(B_4)+r}{\nu(B_4)}\\ =& m(1+r) + \frac{r}{\nu(B_4)}\\ \le& \frac{m(1+r)}{1-r}\end{aligned}$$

4.3. **Construction of horseshoes.** We are going to construct a skew product map with base $\mathcal{G}$ on B_4 and horseshoes on fibers. we mostly follow the argument of A.Katok (for details, see [7] or [2, Section 15.6]). The novelty here is that in our case, for $x \in B_4$ and $z \in V(x)$, $\mathcal{F}(z) = F^{\rho(x)}(z)$ does not necessarily return to $R(g^{\rho(x)}(x)) = R(\mathcal{G}(x))$. To live with this we consider the orbits of g_* and use the admissible rectangles around the points $q(g_*^k(x))$, $k = 1, 2, \ldots, l(x)-1$, to carry over the horseshoe structure until it finally returns to $R(\mathcal{G}(x))$.

For every $x \in B_3$ and $z \in \{x\} \times Y$, let $F_*(z) = F^{N(x)}(z) \in \{g_*(x)\} \times Y$. Note F_* is invertible on $B_3 \times Y$ since F and g_* are invertible. If in addition $z \in P(x)$, then $F_*(z)^{\pm 1} \in P(g_*^{\pm 1}(x))$.

For $x \in B_4$, we note $g_*^k(x) \in B_5$ for $k = 1, 2, \ldots, l(x)-1$ and $g_*^{l(x)}(x) \in B_4$. If $z \in V(x)$, then the connected component

$$CC(R(x) \cap F_*^{-1}(R(g_*(x))), z)$$

is an admissible (s, γ)-rectangle in $R(x)$. As $F_*(z) = F^{N(x)}(z)$ is the $\mathcal{K}$-th return of z to P and $N(x) > m$, and points in $V(x)$ are (d_m^F, ϵ)-separated, from property (5) of the set P, we can conclude that this connected component contains no other points in $V(x)$. So there are $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ such connected components, each of which contains exactly one point in $V(x)$.

Let

$$S_0(z) = CC(R(x) \cap F_*^{-1}(R(g_*(x))), z)$$

and for $k = 1, 2, \ldots, l(x)-1$, define by induction

$$S_k(z) = CC(F_*(S_{k-1}(z)) \cap F_*^{-1}(R(g_*^{k+1}(x))), q(g_*^k(x))) \subset F_*(S_{k-1}(z))$$

Then for each k, $S_k(z)$ is part of an admissible (s, γ)-rectangle in $R(g_*^k(x))$ such that $F_*(S_k(z))$ is an admissible (u, γ)-rectangle in $R(g_*^{k+1}(x))$. Moreover, $F_*^{-(l(x)-1)}(S_{l(x)-1}(z)) \subset S_0(z)$. So we can select for each $z \in V(x)$ a point $u(z) \in F_*^{-(l(x)-1)}(S_{l(x)-1}(z))$. Then

$$\mathcal{F}(u(z)) = F_*^{l(x)}(u(z)) \in F_*(S_{l(x)-1}(z)) \subset R(g_*^{l(x)}(x)) = R(\mathcal{G}(x))$$

and

$$CC(\mathcal{F}(R(x)) \cap R(\mathcal{G}(x)), \mathcal{F}(u(z))) = F_*(S_{l(x)-1}(z)) \subset \mathcal{F}(S_0(z))$$

is an admissible (u, γ)-rectangle in $R(\mathcal{G}(x))$. Note that $\mathcal{F}$ is invertible and $S_0(z)$ are disjoint for different $z \in V(x)$. So there are $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ such rectangles. Likewise, the pre-images

$$\mathcal{F}^{-1}(CC(\mathcal{F}(R(x)) \cap R(\mathcal{G}(x)), \mathcal{F}(u(z)))) = CC(R(x) \cap \mathcal{F}^{-1}(R(\mathcal{G}(x))), u(z))$$

are $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ disjoint admissible (s, γ)-rectangles in $R(x)$.

Let $U(x) = \{u(z) | z \in V(x)\}$. For $x \in B_4$, consider the set

$$\Lambda(x) = \bigcap_{n \in \mathbb{Z}} \mathcal{F}^{-n}\Big(\bigcup_{z' \in U(\mathcal{G}^n(x))} CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z')\Big) \subset R(x)$$

Then $\Lambda(\mathcal{G}(x)) = \mathcal{F}(\Lambda(x))$. $\Lambda(x)$ is the intersection of infinitely many layers, each of which consists of $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ disjoint connected components. If $z'' \in \Lambda(x)$, then for each $n \in \mathbb{Z}$, $\mathcal{F}^n(z'')$ belongs to exactly one of the connected components in

$$\bigcup_{z' \in U(\mathcal{G}^n(x))} CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z')$$

Meanwhile, for any sequence $\{z_n \in U(\mathcal{G}^n(x))\}_{n\in\mathbb{Z}}$, The intersection

$$\bigcap_{n\in\mathbb{Z}} \mathcal{F}^{-n}(CC(R(\mathcal{G}^n(x)) \cap \mathcal{F}^{-1}(R(\mathcal{G}^{n+1}(x))), z_n))$$

contains exactly one point in $\Lambda(x)$. Therefore, $\Lambda = \bigcup_{x\in B_4} \Lambda(x)$ is invariant of $\mathcal{F}$ and $\mathcal{F}|_\Lambda = (\mathcal{G}, \mathcal{H})$, with the base $\mathcal{G}$ on B_4 and $\mathcal{H}$ on the fibers conjugate to the full shift on $[\frac{1}{mr} \exp m(h_\mu(F) - r)]$ symbols.

4.4. **Estimate of entropy.** As $\mathcal{G}$ is ergodic, Λ carries many ergodic invariant measures for $\mathcal{F}|_\Lambda$ of the form

$$\frac{1}{\nu(B_4)} \int_{B_4} \tau_x d\nu$$

where τ_x for each $x \in B_4$ is supported on $\Lambda(x)$ and $\tau_x \circ \mathcal{F}^{-1} = \tau_{\mathcal{G}(x)}$. Entropies of these measures vary from 0 to the topological entropy of the full shift which equals

$$\log[\frac{1}{mr} \exp m(h_\mu(F) - r)]$$

Ergodic measures of arbitrary intermediate entropies can be obtained by properly assigning weights to different symbols for the shift. These measures induce ergodic invariant measures of F. The average return time is

$$\frac{1}{\nu(B_4)} \int_{B_4} \int_{\Lambda(x)} \rho(x) d\tau_x d\nu = \frac{1}{\nu(B_4)} \int_{B_4} \rho(x) d\nu \le m(1+r)/(1-r)$$

So the measures we constructed has the maximal entropy no less than

$$\log[\frac{1}{mr} \exp m(h_\mu(F) - r)] \cdot \frac{1-r}{m(1+r)}$$

which is arbitrarily close to $h_\mu(F)$ as $r \to 0$ and $m \to \infty$. This completes the proof of Main Theorem.

Acknowledgments. The author greatly appreciate the support of Anatole Katok by numerous discussions and encouragement. The author would like to thank the referee for the serious review from which the author learned a lot. Thanks also go to Yuri Kifer who is the editor and kindly explained some issues to the author.

REFERENCES

[1] L. Abramov and V. Rohlin, *Entropy of a skew product of mappings with invariant measure*, (Russian) Vestnik Leningrad. Univ., **17** (1962), 5–13.

[2] L. Barreira and Y. Pesin, "Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents," Cambridge University Press, 2007.

[3] R. M. Belinskaja, *Entropy of a skew product*, (Russian) Izv. Vysš. Učebn. Zaved. Matematika, **128** (1973), 3–14.

[4] M. Herman, *Construction d'un difféomorphisme minimal d'entropie topologique non nulle*, (French) [Construction of a minimal diffeomorphism of nonzero topological entropy], Ergodic Theory Dyn. Syst., **1** (1981), 65–76.

[5] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. I.H.E.S., **51** (1980), 137–173.

[6] A. Katok, *Nonuniform hyperbolicity and structure of smooth dynamical systems*, Proceedings of the International Congress of Mathematicians, Vol. **1, 2** (Warsaw, 1983), 1245–1253, PWN, Warsaw, 1984.

[7] A. Katok and L. Mendoza, *Dynamical systems with nonuniformly hyperbolic behavior*, supplement to "Introduction to the Modern Theory of Dynamical Systems," by A. Katok and B. Hasselblatt, Cambridge University Press, 1995.

[8] J. C. Kieffer and M. Rahe, *The pointwise ergodic theorem for transformations whose orbits contain or are contained in the orbits of a measure-preserving transformation*, Canad. J. Math., **34** (1982), 1303–1318.

[9] P. Sun, *Zero entropy invariant measures for skew product diffeomorphisms*, Ergodic Theory Dyn. Syst., doi:10.1017/S0143385709000376, Published Online by Cambridge University Press, 17 Jul 2009.

Received August 2009; revised February 2010.

E-mail address: peng_sun@math.psu.edu