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Zero-entropy invariant measures for skew product diffeomorphisms

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Abstract. In this paper, we study some skew product diffeomorphisms with non-uniformly hyperbolic structure along fibers and show that there is an invariant measure with zero entropy which has atomic conditional measures along fibers. For such diffeomorphisms, our result gives an affirmative answer to the question posed by Herman as to whether a smooth diffeomorphism of positive topological entropy would fail to be uniquely ergodic. The proof is based on some techniques that are analogous to those developed by Pesin and Katok, together with an investigation of certain combinatorial properties of the projected return map on the base.

1. *Introduction*

Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact s-dimensional smooth manifold M, and let $df : TM \to TM$ be the derivative of f. The diffeomorphism f preserves a Borel probability measure μ. For every x in a set Λ of full measure, the Lyapunov exponent

$$\chi(v, f) = \lim_{n\to\infty} \frac{\ln \|df^n v\|}{n}$$

exists for every non-zero vector $v \in T_x M$. This functional takes at most s values on $T_x M$ and is independent of $x \in \Lambda$ if μ is ergodic. If all Lyapunov exponents are non-zero, then μ is called a hyperbolic measure. Smooth systems with hyperbolic measures are called non-uniformly hyperbolic. The theory for studying such systems was developed by Pesin and then combined with some powerful techniques by Katok to search for invariant orbits, producing a number of profound results. These techniques will serve as cornerstones for our discussion. All the necessary definitions, theorems and background facts relevant to this paper can be found in [**2**] (as a quick reference) or [**3**] (for detailed proofs).

In [**6**], Katok proved the following theorem.

THEOREM 1.1. *Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact manifold M, and let μ be a Borel probability f-invariant hyperbolic measure. Then*

$$\overline{\mathrm{Per}}(f) \supset \mathrm{supp}(\mu)$$

and

$$\max\left(0, \limsup_{n\to\infty} \frac{\ln P_n(f)}{n}\right) \geq h_\mu(f),$$

where $\mathrm{Per}(f)$ *is the set of all periodic points of* f, $P_n(f)$ *is the number of periodic points of* f *with period* n, *and* $h_\mu(f)$ *is the metric entropy with respect to* μ.

In particular, if the manifold M is two-dimensional, then by Ruelle's inequality [**11**] every ergodic invariant measure μ with positive metric entropy must be hyperbolic. Taking the variational principle into account as well, we obtain the following.

COROLLARY 1.2. *For any* $C^{1+\alpha}$ $(\alpha > 0)$ *diffeomorphism* f *of a two-dimensional compact manifold with positive topological entropy,*

$$\limsup_{n\to\infty} \frac{\ln P_n(f)}{n} \geq h(f). \tag{1}$$

Hence f *is not minimal or uniquely ergodic.*

In general, the inequality (1) does not hold in higher-dimensional cases. There can be no periodic orbit for a diffeomorphism with positive topological entropy. Herman [**5**] constructed a remarkable example as follows.

Consider the C^∞ map $A : \mathbb{T}^1 \to SL(2, \mathbb{R})$ defined by

$$A(\theta) = A_\theta = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix},$$

where $\lambda > 1$ is a fixed number. Let $R_\alpha : \mathbb{T}^1 \to \mathbb{T}^1$ be the rotation by $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$.

THEOREM 1.3. (Herman [**5**]) *There is a dense* G_δ *subset* W *of* $\mathbb{T}^1$ *such that for every* $\alpha \in W$, *the smooth diffeomorphism* $F_\alpha = (R_\alpha, A(\theta))$ *on* $\mathbb{T}^1 \times SL(2, \mathbb{R})/\Gamma$ *given by* $(\theta, y) \mapsto (\theta + \alpha, A(\theta) \cdot y)$ *is minimal and has positive topological entropy.*

Herman's example prompted fruitful research, for example, on generic linear cocycles over compact systems. The phenomenon he discovered turned out to be common for $SL(2, \mathbb{R})$ extensions over rotations [**1**].

However, the diffeomorphisms in Herman's example fail to be uniquely ergodic. We can find a measurable transformation $S : \mathbb{T}^1 \to SL(2, \mathbb{R})$ such that for almost every $\theta \in \mathbb{T}^1$,

$$H_\theta = S_{\theta+\alpha} A_\theta S_\theta^{-1} = \begin{pmatrix} l_\theta & 0 \\ 0 & l_\theta^{-1} \end{pmatrix}$$

is diagonal. Thus, for every measure τ preserved by the geodesic flow which corresponds to the left action by

$$G_t = \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix},$$

we have that $\mu_\tau = \int \tau \circ S_\theta \, dm$ is F_α-invariant, where m is the Lebesgue measure on $\mathbb{T}^1$. In particular, if τ is supported on a periodic orbit of the geodesic flow, then $h_{\mu_\tau}(F_\alpha) = 0$.

Whether or not a smooth diffeomorphism of positive topological entropy can be uniquely ergodic is still in question. (For homeomorphisms the answer is yes; see, for example, [**4**].) The author studied some skew product diffeomorphisms and found invariant measures similar to those in Herman's example.

Let (X, m) be a probability measure space and $g : X \to X$ an invertible transformation (mod 0) preserving m. Let M be an l-dimensional compact Riemannian manifold and, for each $x \in X$, let $h_x : M \to M$ be a $C^{1+\alpha}$ diffeomorphism. Assume that $f = (g, h_x)$ on $X \times M$ preserves a measure $\mu = \int \nu_x \, dm$. Let $T_y M = T_p(\{x\} \times M)$ for $y = (x, p) \in X \times M$. In this paper we prove the following result.

THEOREM 1.4. (Main theorem) *If for almost every $y \in Y$ the Lyapunov exponent is such that $\chi(v, f) \neq 0$ for every $v \in T_y M \setminus \{0\}$, then f has an invariant measure whose conditional measure on each fiber is atomic.*

From [**12**, Theorem II], we know that if all Lyapunov exponents along fibers have the same sign, then μ itself must be atomic on each fiber. In this paper we discuss mainly the non-uniformly hyperbolic case where the exponents have different signs.

Now suppose that we have a $C^{1+\alpha}$ diffeomorphism $f = (g, h_x)$ on $X \times M$. Assume that $h(g) = 0$ and that M is two-dimensional. If $h(f) > 0$, we must have $h_\mu(f) > 0$ for some ergodic invariant measure $\mu = \int \nu_x \, dm$. Then, by Ledrappier and Young's formula [**9**], the Lyapunov exponents along the fiber direction must be non-zero almost everywhere. Hence, by Theorem 1.4, f has an invariant measure with atomic conditional measures along fibers. The following statement avoids any mention of exponents.

COROLLARY 1.5. *If f has positive topological entropy, g has zero topological entropy and M is two-dimensional, then f has a measure of zero entropy and is not uniquely ergodic.*

2. *Shadowing lemma*

Now that we have a $C^{1+\alpha}$ diffeomorphism $f = (g, h_x)$ on $Y = X \times M$ that has non-zero exponents along fibers, we may assume that g is ergodic by considering an ergodic component. Almost all of the results in [**6, 8, 10**] can be adapted to this setting with some careful modifications. By considering the derivative $d_y h_x = d_p h_x$ for $y = (x, p)$ as a linear cocycle over f, we have the following theorem.

THEOREM 2.1. *Assume that* $\dim M = l$. *Denote by $B^k(r)$ the standard Euclidean r-ball in $\mathbb{R}^k$ centered at the origin. There exists a set $\Lambda_0 \subset Y$ of full measure such that for every sufficiently small $\epsilon > 0$ and some $\chi > 0$, the following hold.*

(1) *There exist a tempered function $q : \Lambda_0 \to (0, 1]$ and a collection of embeddings $\Psi_y : B^l(q(y)) \to \{x\} \times M$ for each $y = (x, p) \in \Lambda_0$ such that $\Psi_y(0) = y$ and $e^{-\epsilon} < q(y)/q(f(y)) < e^{\epsilon}$.*

(2) *There exist a constant $K > 0$ and a measurable function $C : \Lambda_0 \to \mathbb{R}$ such that for $z_1, z_2 \in B^l(q(y))$,*

$$K^{-1} \, d(\Psi_y(z_1), \Psi_y(z_2)) \leq \|z_1 - z_2\| \leq C(y) \, d(\Psi(z_1), \Psi(z_2)),$$

with $e^{-\epsilon} < C(f(y))/C(y) < e^{\epsilon}$.

(3) *The map* $f_y := \Psi_{f(y)}^{-1} \circ f \circ \Psi_y : B^s(q(y)) \times B^{l-s}(q(y)) \to \mathbb{R}^l = \mathbb{R}^k \times \mathbb{R}^{l-k}$ *has the form*

$$f_y(u, v) = (A_y u + \eta_{1,y}(u, v), B_y v + \eta_{2,y}(u, v)),$$

where $\eta_{1,y}(0, 0) = \eta_{2,y}(0, 0) = 0$, $d\eta_{1,y}(0, 0) = d\eta_{2,y}(0, 0) = 0$ *and*

$$\|A_y\| < \exp -(\chi - \epsilon), \quad \|B_y^{-1}\| < \exp -(\chi - \epsilon).$$

For $z = (u, v) \in B^l(q(y))$ *and* $\eta_y(z) = (\eta_{1,y}(z), \eta_{2,y}(z))$, *we have*

$$\|d_z \eta_y\| < \epsilon, \quad \|\eta_y(z)\| < \epsilon.$$

Definition 2.2. The points $y \in \Lambda_0$ are called regular points. For each regular point y, the set $N(y) = \Psi_y(B(q(y)))$ is called a regular neighborhood of y. Let $r(y)$ be the radius of the maximal ball contained in the regular neighborhood $N(y)$. We say that $r(y)$ is the size of $N(y)$.

THEOREM 2.3. *For each* $\delta > 0$ *and sufficiently small* $\epsilon(\delta) > 0$, *there is a set* $\Lambda_\delta \subset \Lambda_0$ *which has compact intersection* $\Lambda_{\delta,x}$ *(possibly empty) with each fiber* $\{x\} \times M$ *and is such that* $\mu(\Lambda_\delta) > 1 - \delta$ *and the following conditions hold.*

(1) *The functions* $y \mapsto q(y)$, $y \mapsto C(y)$, $y \mapsto \Psi_y$ *(as in Theorem 2.1 with* $\epsilon = \epsilon(\delta)$*) and* $y \mapsto r(y)$ *are all continuous on* $\Lambda_{\delta,x}$ *for each* $x \in X$.

(2) *The decomposition* $T_y M = d_y \Psi_y \mathbb{R}^k \times d_y \Psi_y \mathbb{R}^{l-k}$ *depends continuously on* y *in* $\Lambda_{\delta,x}$.

(3) *On* Λ_δ, *there are bounds* $q_\delta = \min\{q(y)\} > 0$, $r_\delta = \min\{r(y)\} > 0$ *and* $C_\delta = \max\{C(y)\} < \infty$.

With similar definitions and properties for admissible manifolds, we can derive the following version of the shadowing lemma.

THEOREM 2.4. *Given* $\delta > 0$, *for* $\bar{q} < q_\delta$ *set* $\tilde{\Lambda}_\delta(\bar{q}) = \bigcup_{y \in \Lambda_\delta} \Psi_y(B(0, \bar{q}))$. *Given* $a \in \mathbb{Z} \cup \{-\infty\}$ *and* $b \in \mathbb{Z} \cup \{\infty\}$, *a sequence* $\{y_n = (x_n, p_n)\}_{a<n<b}$ *is called a* $(\delta, \bar{q})$*-pseudo-orbit for* $f = (g, h_x)$ *if there are* $\{z_n \in \Lambda_{\delta,x_n}\}_{a<n<b}$ *and* $\{k_n\}_{a<n<b}$ *such that for every* n *we have* $y_n \in \Psi_{z_n}(B(0, \bar{q}))$ *and* $f^{k_{n+1}-k_n}(y_n) \in \Psi_{z_{n+1}}(B(0, \bar{q}))$. *Then there exists* $\gamma = \gamma(\delta)$ *such that for every* (δ, γ)*-pseudo-orbit, there is a unique point* $\tilde{y} \in Y$ *such that* $f^{k_n}(\tilde{y}) \in \Psi_{z_n}(B(0, q_\delta))$ *for all* $a < n < b$.

3. *Integrability of the return time*

Now we would like to take a proper Pesin set on which the shadowing techniques can be carried out.

Definition 3.1. Let $\pi : Y \to X$ be the projection to the base. A measurable subset $P \subset Y$ is called a 'regular tube' if for some $\delta > 0$, $\epsilon > 0$, $\nu_0 > 0$ and $\gamma = \gamma(\delta)$ as in Theorem 2.4, there exists for each $x \in B = \pi(P)$ a point $z(x) \in \Lambda_{\delta,x}$ such that $P_x = P \cap (\{x\} \times M) \subset \Psi_{z(x)}(B(0, \gamma))$, $\nu_x(P_x) > \nu_0$ and $m(B) > 1 - \epsilon$.

The existence of such a 'regular tube' was guaranteed in §2. In this regular tube, we can take a measurable section $s : B \to P$ such that $\pi \circ s = \mathrm{id}_B$. Let $S = s(B)$. We shall then consider the first return map f_P on P.

PROPOSITION 3.2. *The map s can be chosen in such a way that the first return time from S to P is integrable with respect to m. In particular, we may assume that every point in S returns to P in finite time.*

Proof. For each $y \in P$, denote by $n(y)$ the return time of y. Since μ is f-invariant and $\mu(P) > m(B) \cdot \nu_0 > 0$, we have

$$0 < \int_P n(y)\, d\mu = \mu\bigg(\bigcup_{j\geq 0} F^j(P)\bigg) \leq 1.$$

But

$$\int_P n(y)\, d\mu = \sum_{j=0}^{\infty} \mu(P_j),$$

where $P_j = P\backslash(\bigcup_{1\leq k\leq j} F^{-k}(P))$.

We may choose s such that for every $x \in B$, $s(x) \in P_j$ only if $\nu_x(P_x \backslash P_j) = 0$. Let $B_j = \pi_1(S\backslash(\bigcup_{1\leq k\leq j} F^{-k}(P)) = \pi_1(S \cap P_j)$. Because of the way s was chosen and the assumption that $\nu_x(P_x) > \nu_0$, we have $\mu(P_j) \geq \mu(\pi_1^{-1}(B_j) \cap P) > m(B_j) \cdot \nu_0$; hence

$$\int_B n(s(x))\, dm = \sum_{j=0}^{\infty} m(B_j) < \sum_{j=0}^{\infty} \frac{1}{\nu_0}\mu(P_j) \leq \frac{1}{\nu_0}.$$

Thus the return time is integrable. □

4. *Projected return map on the base*

Now let us try to find the invariant measure described in Theorem 1.4. We may assume that g has no periodic point, or else the problem can be reduced to the case considered by Katok [**6**]. Moreover, g is invertible, as assumed earlier.

We are looking for an invariant set I which has finite intersection I_x with almost every fiber $\{x\} \times M$. The measure τ_x supported on I_x is the delta counting measure on $\{x\} \times M$. Then an invariant measure for f can be given by $\int \tau_x\, dm$.

To find the invariant set I, we start with a fixed 'regular tube' P and a measurable section s as specified in §3. Let f_P be the first return map on P and g_B the first return map for g on B. Define $r : B \to B$ by $r(x) = \pi \circ f_P(s(x))$; then $r(x)$ is the projection of the return map on the base. Note that g_B is invertible but r may not be. For each $x \in B$, let $k(x)$ be such that $f^{k(x)} = f_P(s(x))$.

We define a partial order on B as follows: $x_1 \prec x_2$ if and only if there is $n \geq 0$ such that $g_B^n(x_1) = x_2$, i.e. x_2 is an image of x_1 under iterates of g. Since g is invertible and has no periodic point, this partial order is well-defined. Moreover, if there is $n \geq 0$ such that $r^n(x_1) = x_2$, then we write $x_1 \prec\!\prec x_2$, which implies that $x_1 \prec x_2$. This is also a partial order.

We now define an equivalence relation on B: we say that $x_1 \backsim x_2$ if and only if

$$Q(x_1, x_2) := \{x \in B \mid x_1 \prec\!\prec x \text{ and } x_2 \prec\!\prec x\} \neq \emptyset,$$

i.e. there are $n_1, n_2 > 0$ such that $r^{n_1}(x_1) = r^{n_2}(x_2)$. If $x \in Q(x_1, x_2)$, then $r^n(x) \in Q(x_1, x_2)$ for all $n > 0$. If $x_1 \backsim x_2$, we must have $x_1 \prec x_2$ or $x_2 \prec x_1$, which we write as $x_1 \precsim x_2$ or $x_2 \precsim x_1$. If $x_1 \precsim x_2$, define $\sigma(x_1, x_2)$ as the minimal (with respect to $\prec$ throughout this paper) element in $Q(x_1, x_2)$. In particular, if $x_1 \prec\prec x_2$, then $x_1 \precsim x_2$ and $\sigma(x_1, x_2) = x_2$.

Remark. The equivalence relation $\sim$ defined here is crucial. If $x_1 \precsim x_2$, then $s(x_1)$ and $s(x_2)$ return to the same fiber after iteration of f_P. However, $s(x_1)$ does not necessarily return to P_{x_2}, i.e. two points in S may return to P on the same fiber and this could happen all the time. The author had trouble dealing with this situation while looking for pseudo-orbits; introduction of the above equivalence relation solved the problem. We can then, in each equivalence class, find a unique orbit of r (a sequence of returns, lifted to a pseudo-orbit) to construct the invariant set.

PROPOSITION 4.1. *For almost every $x \in B$, $J(x) = \{\sigma(x', x) \mid x' \precsim x, x' \neq x\}$ is finite. Denote by x^* the maximal element of $J(x)$. Then $x' \prec\prec x^*$ for all $x' \precsim x$. Let $W(x) = \{\bar{x} \mid x' \prec\prec \bar{x}$ for all $x' \precsim x\}$; then $x^* = \min W(x)$. Moreover, if $x_1 \precsim x_2$, then $x_1^* \prec\prec x_2^*$.*

Proof. For every $x \in B$, define the set of 'jumps' $J'(x) := \{r(x') \mid x' \prec x, x' \neq x$ and $x \prec r(x')\}$. By integrability of return times (Proposition 3.2), $J'(x)$ must be finite for almost every $x \in B$. To see this, consider the set

$$\tilde{S} = \bigcup_{x \in B} \{f(s(x)), f^2(s(x)), \ldots, f^{k(x)}(s(x))\}.$$

Let $\tilde{S}_j = \{x \in X : |\tilde{S} \cap (\{x\} \times M)| = j\}$. We can count the return times and get

$$\sum_{j=0}^{\infty} j \cdot m(\tilde{S}_j) = \int_B k(x)\, dm < \infty$$

and that $|J'(x)| \leq |\tilde{S} \cap (\{x\} \times M)| < \infty$ for almost every $x \in B$.

For every $x' \precsim x$ with $x' \neq x$, there must be $\bar{x} \in J'(x)$ such that $x' \prec\prec \bar{x}$ and $\sigma(x', x) = \sigma(x, \bar{x})$. So, for different elements $x_1, x_2 \in J(x)$, there must be different elements $\bar{x}_1, \bar{x}_2 \in J'(x)$ such that $\sigma(\bar{x}_i, x) = x_i$ for $i = 1, 2$. Hence $|J(x)| \leq |J'(x)| < \infty$.

By definition, $x' \prec\prec \sigma(x', x)$ for every $x' \in B$, and $\sigma(x_1, x) \prec\prec \sigma(x_2, x)$ if $\sigma(x_1, x) \prec \sigma(x_2, x)$ since they are both images of x under iteration of r. So for every $x' \precsim x$, we have $x' \prec\prec \sigma(x', x) \prec\prec x^*$.

Since $x^* \in J(x)$, there is some $x' \precsim x$ such that $\sigma(x', x) = x^*$. Then every $\bar{x} \in W(x)$ satisfies $\bar{x} \in Q(x', x)$. But $x^* = \sigma(x', x) = \min Q(x', x) \prec \bar{x}$; thus $x^* = \min W(x)$.

If $x_1 \precsim x_2$, then

$$x_1^* = \min\{x \mid x' \prec\prec x \text{ for all } x' \precsim x_1\} \prec \min\{x \mid x' \prec\prec x \text{ for all } x' \precsim x_2\} = x_2^*,$$

because the second set is contained in the first one. But $x_1 \prec\prec x_1^*$ and $x_1 \prec\prec x_2^*$, so from the previous discussion we must have $x_1^* \prec\prec x_2^*$. □

PROPOSITION 4.2. *Let $B_0 = \{x \in B \mid$ there is no $x' \neq x \in B$ such that $x' \precsim x\}$. Then $m(B_0) = 0$. Hence, upon replacing B by $B \backslash (\bigcup_{k \in \mathbb{Z}} g_B^k(B_0))$ and P accordingly, we may assume that for every $x \in B$ there is at least one element $x' \in B$ such that $x' \precsim x$ but $x' \neq x$.*

Proof. If $m(B_0) > 0$, then by the Poincaré recurrence theorem there must be an element $x_0 \in B_0$ such that $B_0(x_0) = \{g_B^{-n}(x_0), n \in \mathbb{N}\} \cap B_0$ has infinitely many elements, because g_B is invertible and m-preserving. From the proof of Proposition 4.1, $J'(x_0)$ has finitely many elements. However, for every $x \in B_0(x_0)$, there must be $x' \in J'(x_0)$ such that $x \prec\prec x'$. Hence there is an element $\tilde{x} \in J'(x_0)$ such that $\tilde{B}_0(x_0) = \{x \in B_0(x_0) \mid x \prec\prec \tilde{x}\}$ has infinitely many elements. But $x_1 \sim x_2$ for all $x_1, x_2 \in \tilde{B}_0(x_0) \subset B_0$ because $\bar{x} \in Q(x_1, x_2) \neq \emptyset$, which is a contradiction. □

For each $x \in B$, let $G(x) := \{x' \in B \mid x' \sim x\}$ and $G^*(x) := \{(x')^* \mid x' \in G(x)\}$. If $x_1 \sim x_2$, then we must have $G(x_1) = G(x_2)$ and $G^*(x_1) = G^*(x_2)$.

We pick $H(x)$ in the following way.

(1) If $G^*(x)$ is not properly defined, let $H(x) = \emptyset$ (only for x in a set of measure zero).

(2) If $G^*(x)$ has a minimal element $\tilde{x}$, let $H(x) = \{f^n(s(\tilde{x}))\}_{-\infty<n<\infty}$.

(3) If $G^*(x)$ has no minimal element, then by Proposition 4.1 $G^*(x)$ can be completed to a full orbit of r, $\bar{G}(x) = \bigcup_{0\le n<\infty} r^n(G^*(x))$. In each equivalence class $G(x)$, $\bar{G}(x)$ is a sequence of returns and is uniquely defined in the following sense:

$$\bar{G}(x) = \bigcup_{x_1 \sim x} \bigcap_{x_2 \precsim x_1} \{r^n(x_2)\}_{0\le n<\infty}.$$

Thus, $\bar{G}(x)$ ordered by '$\prec\prec$' can be viewed as a sequence $\{\bar{x}_n\}_{-\infty<n<\infty}$, and $r(x_n) = x_{n+1}$ for all n. The sequence $\{s(\bar{x}_n)\}_{-\infty<n<\infty}$ is therefore in fact a (δ, γ)-pseudo-orbit; let us call it the pseudo-orbit associated to x. Note that pseudo-orbits associated to equivalent elements coincide. Because of how the 'regular tube' P was chosen, we can find $\tilde{y} \in Y$ as specified in Theorem 2.4. Let $H(x) = \{f^n(\tilde{y})\}_{-\infty<n<\infty}$.

Let $I = \bigcup_{x\in B} H(x)$. By definition, $H(x)$ is invariant for all $x \in B$. Hence I is f-invariant.

PROPOSITION 4.3. *For almost every $x \in X$, $I_x = I \cap (\{x\} \times M)$ is non-empty and contains finitely many elements.*

Proof. For almost every $x \in B$, $I_x \supset (H(x) \cap (\{x\} \times M)) \neq \emptyset$ by definition. Note that $H(x_1) = H(x_2)$ if $x_1 \sim x_2$. For different elements $y_1, y_2 \in I_x$, there are $x_1, x_2 \in B$ such that $y_i \in H(x_i)$ for $i = 1, 2$. We must have $x_1 \nsim x_2$ and $G(x_1) \cap G(x_2) = \emptyset$. But $G(x_i) \cap J'(x) \neq \emptyset$ for $i = 1, 2$. So $|I_x| \le |J'(x)|$.

Recall that $m(B) > 0$; therefore, by ergodicity, I_x must be non-empty and contain finitely many elements for almost every $x \in X$. □

Since I is invariant and I_x is finite, $\int \tau_x \, dm$ is an invariant measure as desired, where τ_x is the delta counting measure on $\{x\} \times M$ supported on I_x, for almost every $x \in X$. The entropy of this measure is the same as the entropy of the transformation g on the base.

5. *Measures of intermediate entropies*

In [**7**], Katok showed a stronger result.

THEOREM 5.1. *If $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold and μ is an ergodic hyperbolic measure for f with $h_\mu(f) > 0$, then for any $\epsilon > 0$ there* *exists a hyperbolic horseshoe Γ such that $h(f|_\Gamma) > h_\mu(f) - \epsilon$. Hence, for any number β between zero and $h_\mu(f)$, there is an ergodic invariant measure μ_β such that $h_{\mu_\beta(f)} = \beta$.*

The detailed proof of this theorem can be found in [**8**] and [**3**]. We seek an analogous result for our skew product diffeomorphisms, where μ is not necessarily a hyperbolic measure but the neutral direction can be factored out in such a way that zero exponents appear only in the base. We would like to show that there are ergodic measures with arbitrary intermediate entropy between the entropy on the base and the entropy of the skew product. In this case we would not expect to have any proper closed invariant subset; however, we can find an invariant set Γ that has closed intersection with almost every fiber, on which f acts like a horseshoe map. Moreover, this horseshoe can carry an entropy arbitrarily close to $h_\mu(f)$, to produce invariant measures with arbitrary intermediate entropies. This work is in progress [**13**].

One might also ask the question of whether Theorem 5.1 holds for any $C^{1+\alpha}$ diffeomorphism without the assumption that μ is hyperbolic. In this general case, not even a zero-entropy measure has yet been found.

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